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AN EXTENSION THEOREM FOR SEPARABLE BANACH SPACES

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ABSTRACT

We construct a totally disconnected ω^*-closed, norming subset F of the unit ball B^* of an arbitrary separable Banach space, X, and an operator from $C(F)$ to $C(B^*)$ that "almost" commutes with the natural embeddings of X. This is used to give a new proof of Milutin's theorem and to prove some new results on complemented subspaces of $C[0,1]$ with separable dual. In particular we show that a complemented subspace of $C(\omega^\omega)$, is either isomorphic to $C(\omega^\omega)$ or to c_0.

§1. Introduction

Let X be a Banach space and F a ω^*-closed subset of B^*—the unit ball of X^*. The set F is called λ-norming if $\sup\{\theta(x): \theta \in F\} \geqq \lambda \|x\|$ for all $x \in X$. If F is a ω^*-closed subset of B^*, we denote by i_F the natural map of X into $C(F)$ (the space of continuous functions on F) defined by $(i_F x)(\theta) = \theta(x)$. The set F is λ-norming iff i_F is an isomorphism into $C(F)$ satisfying $\lambda \|x\| \leqq \|i_F x\| \leqq \|x\|$ for every $x \in X$. In the special case that $F = B^*$, we shall denote i_{B^*} by i. Clearly i is an isometry.

The main result of this paper is the following extension theorem.

THEOREM 1. *Let X be a separable Banach space and $\varepsilon > 0$. Then there exists a totally disconnected, ω^*-closed, $(1-\varepsilon)$-norming subset F of B^*, and a norm one operator $S: C(F) \to C(B^*)$ satisfying $\|Si_F x - ix\| \leqq \varepsilon \|x\|$ for all $x \in X$.*

To explain the theorem, assume (although this is impossible in general) that $\varepsilon = 0$. Then the claim would be that one can find a totally disconnected, ω^*-closed 1-norming set F, and a norm one operator $S: C(F) \to C(B^*)$ for which the following diagram commutes:

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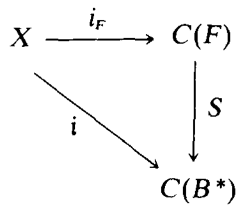

With the ε, the theorem says that this can "almost" be done.

Theorem 1 will be proved in Section 3. In Section 2 we give some applications of this theorem. The first is a new proof of Milutin's theorem on the isomorphism of the spaces $C(K)$, K uncountable compact metric.

The second application is to the study of complemented subspaces of $C(0,1)$ with separable dual. We use the recent profound results of M. Zippin [11], and prove the following:

THEOREM 2. *Let X be isomorphic to a complemented subspace of $C(0,1)$, with X^* separable. Then there exists a countable compact metric space K, such that X is a quotient of $C(K)$.*

It is well known [7] that for every countable metric space K, there is a countable ordinal α, such that K is homeomorphic to $\{\beta : \beta \leqq \alpha\}$, where the latter is equipped with the order topology. We denote by $C(\alpha)$ the space of continuous functions on $\{\beta : \beta \leqq \alpha\}$.

Bessaga and Pelczynski, [3], proved that if $\alpha \leqq \beta$, then $C(\alpha)$ is isomorphic to $C(\beta)$ iff $\beta < \alpha^\omega$. In particular the first α such that $C(\alpha)$ is not isomorphic to c_0 is ω^ω. Theorem 2 is, in fact, quantitative, in the sense that it gives the α for which X is a quotient of $C(\alpha)$. This fact, together with a recent result of Alspach [1], are used to prove our third theorem.

THEOREM 3. *Let X be isomorphic to a complemented subspace of $C(\omega^\omega)$. Then X is either isomorphic to c_0 or to $C(\omega^\omega)$.*

We shall use standard Banach space terminology (see e.g. [6]). All the results apply to real and complex Banach spaces.

§2. Applications

For a Banach space X, let $\lambda(X) = \inf\{\|T\|\,\|T^{-1}\|\,\|P\|\}$, where the inf is taken over all isomorphisms T from X into a $C(K)$ space, and P is a projection from $C(K)$ onto TX. It was proved in [2] that $\lambda(X) = \inf\{\|Q\|\}$, where the inf is taken over all projections Q from $C(B^*)$ onto iX.

We can now formulate the consequences of Theorem 1 in the form in which they will be used.

PROPOSITION 1. *Let X be a separable Banach space with $\lambda(X) < \infty$. If $\varepsilon < (3\lambda(X))^{-1}$, there is a ω^*-closed, $(1-\varepsilon)$-norming totally disconnected subset of B^*, and a projection P from $C(F)$ onto $i_F X$ with $\| P \| \leqq 2\lambda(X)$.*

PROOF. Since $\lambda(X) < \infty$ there is a projection $Q : C(B^*) \to iX$. By Theorem 1 there are F and S satisfying the conclusion of the theorem for the given ε. Define $T : C(F) \to i_F X$ by $T = i_F i^{-1} QS$. Then $\| T \| \leqq \| Q \|$, and $\| Ti_F x - i_F x \| \leqq \varepsilon(1-\varepsilon)^{-1} \| Q \| \, \| i_F x \|$, for every $x \in X$. Indeed, since $Qix = ix$ we have

$$\| Ti_F x - i_F x \| = \| i_F i^{-1} QSi_F x - i_F x \| \leqq \| i_F \| \| i^{-1} \| \| Q \| \| Si_F x - ix \| \leqq \varepsilon \| Q \| \| x \|.$$

Since F is $(1-\varepsilon)$ norming $\| x \| \leqq (1-\varepsilon)^{-1} \| i_F x \|$.

By a standard argument, if $\varepsilon(1-\varepsilon)^{-1}\| Q \| < 1$ the operator $T|_{i_F X}$ is an isomorphism of $i_F X$ onto itself, with inverse W satisfying $\| W \| \leqq \Sigma(\varepsilon(1-\varepsilon)^{-1} \| Q \|)^n$. The desired projection is $P = WT$, and its norm satisfies $\| P \| \leqq \| W \| \, \| T \| \leqq 2\lambda(X)$, if $\| Q \|$ is close enough to $\lambda(X)$.

COROLLARY (Milutin's theorem). *If K is an uncountable, compact metric space, then $C(K)$ is isomorphic to $C(0,1)$.*

PROOF. By standard arguments (see e.g. [9]), it is enough to show that $C(K)$ is isomorphic to a complemented subspace of $C(F)$ for some totally disconnected, compact, metric space F. But $X = C(K)$ certainly satisfies the conditions of Proposition 1. Thus the result follows if we use the space F, the isomorphism i_F and the projection P, guaranteed by this proposition.

REMARKS. (1) For the special case $X = C(K)$ above, it is very easy to write explicitly a norm one projection $Q : C(B^*) \to iX$. Namely, for $f \in C(B^*)$ and $k \in K$ define $(i^{-1}Qf)(k) = f(\delta_k)$ (where $\delta_k \in B^*$ is the point evaluation given by $\delta_k(x) = x(k)$ for all $x \in C(K)$).

(2) For other proofs of Milutin's theorem, see [8], [4], [9].

We now pass to our second application, but we first need some notation.

Let X be a Banach space with a separable dual, and $\delta > 0$. Let H be a ω^*-closed subset of B^*. The first δ-Szlenk subset H_1 of H is defined as follows:

$$H_1 = \left\{ \begin{array}{c} \theta \in B^* : \exists(\theta_n) \subseteq H, \exists(x_n) \subseteq X \quad \text{such that} \quad \theta_n \xrightarrow{\omega^*} \theta,\ x_n \xrightarrow{\omega} 0, \\ \| x_n \| \leqq 1 \quad \text{and} \quad \limsup \theta_n(x_n) \geqq \delta. \end{array} \right\}$$

Szlenk [10] has proved that H_1 is a closed subset of H. We can now define inductively for all ordinals γ:

$$H_{\gamma+1} = (H_\gamma)_1, \qquad \text{and} \qquad H_\gamma = \bigcap_{\beta<\gamma} H_\beta \quad \text{if } \gamma \text{ is a limit ordinal.}$$

Szlenk has also shown that there is a countable ordinal $\eta = \eta(H, \delta)$ such that $H_\eta \neq \emptyset$ and $H_{\eta+1} = \emptyset$. We shall denote by $\eta(X, \delta)$ the ordinal obtained for $H = B^*$. $\eta(X, \delta)$ is called the δ-Szlenk index of X. Obviously $\eta(H, \delta) \leqq \eta(X, \delta)$ for all $H \subset B^*$.

We can now formulate Zippin's result [11]. Although the following lemma is not formulated as such in [11], it summarizes Zippin's beautiful construction.

ZIPPIN'S LEMMA. *Let X be a Banach space with separable dual, and $\varepsilon > 0$. Let F be a ω^*-closed totally disconnected, $(1-\varepsilon)$-norming subset of B^*. Then there is a countable ordinal α, $\alpha < \omega^{(\eta(X,\varepsilon/8)+1)}$, and a subspace Y of $C(F)$, isometric to $C(\alpha)$, such that for every $x \in X$, there is a $y \in Y$ with $\| i_F x - y \| \leqq \varepsilon(1-\varepsilon)^{-1} \| i_F x \|$.*

PROOF OF THEOREM 2. Since $\lambda(x) < \infty$, take $\varepsilon < (10\lambda(x))^{-1}$, and by Proposition 1, find a totally disconnected, ω^*-closed $(1-\varepsilon)$-norming subset F of B^*, with a projection $P: C(F) \to i_F X$ satisfying $\| P \| \leqq 2\lambda(X)$.

By Zippin's lemma, there is a subspace Y of $C(F)$, isometric to $C(\alpha)$, such that for every $x \in X$ there is a $y \in Y$ with $\| i_F x - y \| \leqq \varepsilon(1-\varepsilon)^{-1} \| i_F x \|$.

Since $\varepsilon(1-\varepsilon)^{-1} \| P \| < 1$, the restriction of P to Y maps Y *onto* $i_F X$. Thus X is a quotient of $C(\alpha)$.

PROOF OF THEOREM 3. We distinguish the two cases by considering the Szlenk index of X. By a theorem of Alspach [1], if there is a $\delta > 0$ for which $\eta(X, \delta) \geqq \omega$, then X contains a complemented subspace isomorphic to $C(\omega^\omega)$. Thus X is isomorphic to $C(\omega^\omega)$ by Pelczynski's decomposition method [6].

If $\eta(X, \delta) < \omega$ for every $\delta > 0$, then by the proof of Theorem 2 and Zippin's lemma, X is a quotient of $C(\alpha)$ for $\alpha < \omega^{m+1}$ where $m = \eta(X, \varepsilon/8) < \omega$. By [3] $C(\alpha)$ is isomorphic to c_0, and thus X, being an $\mathscr{L}_\infty$ quotient of c_0, is isomorphic to c_0 by [5].

§3. Proof of Theorem 1

Let $F_0 = \{\theta \in X^* : \| \theta \| \leqq 1 - \varepsilon/2\}$. We shall construct a ω^*-closed, totally-disconnected, $(1-\varepsilon)$-norming subset F of B^*, and a norm one operator $T: C(F) \to C(F_0)$ satisfying $\| T i_F x - i_{F_0} x \| \leqq \frac{1}{2}\varepsilon \| x \|$. We then define

$W: C(F_0) \to C(B^*)$ by $(Wf)(\theta) = f((1-\varepsilon/2)\theta)$, and $S = WT$. Obviously $\| S \| \leqq 1$, and for every $x \in X$ and $\theta \in B^*$

$$|(Si_F x)(\theta) - ix(\theta)| \leqq |(Ti_F x)((1-\varepsilon/2)\theta) - i_{F_0}x((1-\varepsilon/2)\theta)| + \tfrac{1}{2}\varepsilon\,|x(\theta)|$$

$$\leqq \tfrac{1}{2}\varepsilon \| x \| + \tfrac{1}{2}\varepsilon \| x \| = \varepsilon \| x \|.$$

Since X is separable, we can find a biorthogonal sequence (x_n, θ_n) in X with $\| x_n \| = 1$ for all n, such that $\{x_n\}$ span a dense subset of X. The metric $\rho(\theta, \psi) = \Sigma 2^{-(n+1)} | \theta(x_n) - \psi(x_n) |$ induces the ω^*-topology on B^*.

For $n \geqq 1$, let M_n be integers satisfying $\Sigma \| \theta_n \| M_n^{-1} < \varepsilon/2$, and $A_n = \{ j/M_n : j$ is an integer, $|j| \leqq M_n\}$. Let $F_n = \{\theta \in B^* : \theta(x_j) \in A_j\ j = 1, \cdots, n$ and $\| \theta \| \leqq 1 - \Sigma_{i>n} \| \theta_i \| M_i^{-1}\}$ and set $F = \bigcap_{n>1} \overline{(\bigcup_{j \geqq n} F_j)}^{\omega^*}$.

Every point $\theta \in F$ satisfies $\theta(x_n) \in A_n$ for all n, thus the map $\phi : F \to \Pi_{n \geqq 1} A_n$, defined by $\phi(\theta) = (\theta(x_1), \theta(x_2), \cdots)$, is a homeomorphism of F into the Cantor set ΠA_n. Consequently F is totally disconnected.

To show that F is $(1-\varepsilon)$-norming, we shall show that it is an $\varepsilon/2$-net (in the norm topology!) in $(1-\varepsilon/2)B^*$. Thus let $\| \psi \| \leqq 1 - \varepsilon/2$, and for each n, choose j_n such that $(j_n - 1)M_n^{-1} \leqq \psi(x_n) \leqq (j_n + 1)M_n^{-1}$, and define $\theta = \psi + \Sigma_{n \geqq 1} (j_n M_n^{-1} - \psi(x_n))\theta_n$. Since $\Sigma \| (j_n M_n^{-1} - \psi(x_n))\theta_n \| \leqq \Sigma M_n^{-1} \| \theta_n \| < \varepsilon/2$, θ is well defined, $\| \theta - \psi \| < \varepsilon/2$, and $\| \theta \| < 1$. Since also $\theta(x_n) = j_n M_n^{-1} \in A_n$ for all n, the condition $\| \theta \| < 1$ implies that $\theta \in F_n$ for all large enough n, hence $\theta \in F$.

To construct T, we shall use the fact that the points in F_{n-1} are "very close" to points in F_n, to construct for each $n \geqq 1$ a norm one operator $T_n : C(F_n) \to C(F_{n-1})$ which almost commutes with the embeddings of X (see (b) below). The operator T is then obtained from the T_n's by a limiting procedure.

Thus fix $n \geqq 1$. We shall construct $T_n : C(F_n) \to C(F_{n-1})$, satisfying:

(a) For every $\theta \in F_{n-1}$, $(T_n f)(\theta)$ is a convex combination of the values of f at some points $\{\psi_j\}$ in F_n (depending on θ) with $\rho(\theta, \psi_j) \leqq 2^{-n}$ for every j.

(b) $\| T_n i_{F_n} x - i_{F_{n-1}} x \| \leqq M_n^{-1} \| \theta_n \| \; \| x \|$.

Let $\{g_j(t)\}$ be a partition of unity for $[-1, 1]$, with $\{t : g_j(t) \neq 0\} \subset ((j-1)M_n^{-1}, (j+1)M_n^{-1})$, and define $T_n(f)$ for $f \in C(F_n)$ and $\theta \in F_{n-1}$ by

$$(T_n f)(\theta) = \sum_{|j| \leqq M_n} g_j(\theta(x_n)) f(\theta + (jM_n^{-1} - \theta(x_n))\theta_n).$$

Also let $\psi_j = \theta + (jM_n^{-1} - \theta(x_n))\theta_n$. If for some j, $g_j(\theta(x_n)) \neq 0$, then $(j-1)M_n^{-1} < \theta(x_n) < (j+1)M_n^{-1}$, and thus $\| \theta - \psi_j \| \leqq M_n^{-1} \| \theta_n \|$. Also, since $\theta \in F_{n-1}$, $\psi_j(x_i) \in A_i$ for all $i \leqq n$.

Thus T_n is well defined: If the j^{th} term in the sum is non-zero, then

$\| \psi_j \| \leqq \| \theta \| + M_n^{-1} \| \theta_n \| \leqq 1 - \Sigma_{i>n} M_i^{-1} \| \theta_i \|$ (because $\theta \in F_{n-1}$), and thus $\psi_j \in F_n$. (It is also clear that $T_n f$ is a continuous function.)

Since for every θ, the $g_j(\theta(x_n))$ are non-negative and sum to one, $(T_n f)(\theta)$ is a convex combination of the numbers $f(\psi_j)$. Also $\rho(\psi_j, \theta) = 2^{-(n+1)} | \theta(x_n) - \psi_j(x_n) | \leqq 2^{-n}$ for every j for which $g_j(\theta(x_n)) \neq 0$, and thus (a) holds. (Note that (a) also implies that $\| T_n \| \leqq 1$.)

To prove (b), notice that if $x \in X$, it is linear on B^*, hence

$$(T_n i_{F_n} x)(\theta) = \theta(x) + \Sigma g_j(\theta(x_n))(jM_n^{-1} - \theta(x_n))\theta_n(x),$$

or (since $(i_{F_{n-1}} x)(\theta) = \theta(x)$)

$$| (T_n i_{F_n} x)(\theta) - (i_{F_{n-1}} x)(\theta) | \leqq M_n^{-1} \| \theta_n \| \, \| x \|$$

(because each of the non-zero terms in the convex combination is bounded by the right hand side).

The limiting procedure is described in the following:

Claim. Let $f \in C(F)$, and let h be any continuous extension of f to B^*. Then $f_n = (T_1 \cdots T_n)(h \mid_{F_n})$ is a Cauchy sequence in $C(F_0)$. Moreover, $Tf = \lim f_n$ is independent of the particular choice of the extension h.

Proof of the Claim. Fix any $\delta > 0$. By the continuity of h there is an $\eta > 0$ such that $\rho(\theta, \psi) < \eta$ implies that $| h(\theta) - h(\psi) | < \delta$, and choose N so that $\Sigma_{n \geqq N} 2^{-n} < \eta$. Since $\| T_j \| = 1$, we get for every $n > m \geqq N$ that

$$\| f_n - f_m \| = \| T_1 \cdots T_m(h \mid_{F_m}) - T_1 \cdots T_n(h \mid_{F_n}) \|$$
$$\leqq \| h \mid_{F_m} - T_{m+1} \cdots T_n(h \mid_{F_n}) \| .$$

Thus fix any $\theta \in F_m$. Using (a) inductively, we see that $(T_{m+1} \cdots T_n(h \mid_{F_n}))(\theta)$ is a convex combination of the values of h at some points $\{\psi_j\}$ in F_n, with $\rho(\theta, \psi_j) \leqq \Sigma_m^n 2^{-k} < \eta$. But then $| h(\theta) - h(\psi_j) | < \delta$ for all j, and the same holds for their convex combinations, i.e., $| (T_{m+1} \cdots T_n(h \mid_{F_n}))(\theta) - h(\theta) | < \delta$, and $\{f_n\}$ is indeed a Cauchy sequence.

Also if h_1 is another extension of f to B^*, let $U = \{\theta : | h(\theta) - h_1(\theta) | < \delta\}$. U is an open neighborhood of F, and thus, by the definition of F, there is an N so that $F_n \subset U$ for every $n \geqq N$. Using again the fact that $\| T_n \| = 1$ for all n, we get that $\| T_1 \cdots T_n(h \mid_{F_n} - h_1 \mid_{F_n}) \| \leqq \delta$.

The operator $T : C(F) \to C(F_0)$ defined by the claim is the desired operator. It is obviously linear, and $\| T \| = 1$ by an argument similar to the last argument in the proof of the claim.

Now if $x \in X$, we can take $h = ix$ as the extension of $f = i_F x$. We then get by (b) that for every $n \gtrsim 1$

$$\| T_1 \cdots T_n i_{F_n} x - i_{F_0} x \| \leqq \sum_j \| T_1 \cdots T_j i_{F_j} x - T_1 \cdots T_{j-1} i_{F_{j-1}} x \|$$
$$\leqq \Sigma \| T_j i_{F_j} x - i_{F_{j-1}} x \| \leqq \Sigma M_j^{-1} \| \theta_j \| \| x \|$$
$$\leqq \varepsilon/2 \| x \| .$$

Hence also in the limit $\| Ti_F x - i_{F_0} x \| \leqq \varepsilon/2 \| x \|$.

REMARK. It may be of some importance to note that the operator T is "almost" a simultaneous extension operator. Indeed, one easily checks that if $\theta \in F_n \cap F_{n-1}$ then $(T_n f)(\theta) = f(\theta)$ for all $f \in C(F_n)$. And thus also in the limit we get that $Tf(\theta) = f(\theta)$ for all $\theta \in F \cap F_0$ and $f \in C(F)$.

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